

## FIELD THEORIES ON THE POINCARÉ DISK

FRANCO FERRARI

*Dipartimento di Fisica, Università di Trento, 38050 Povo (TN), Italy  
and INFN, Gruppo Collegato di Trento, Italy*

## ABSTRACT

The massive scalar field theory and the chiral Schwinger model are quantized on a Poincaré disk of radius  $\rho$ . The amplitudes are derived in terms of hypergeometric functions. The behavior at long distances and near the boundary of some of the relevant correlation functions is studied. The exact computation of the chiral determinant appearing in the Schwinger model is obtained exploiting perturbation theory. This calculation poses interesting mathematical problems, as the Poincaré disk is a noncompact manifold with a metric tensor which diverges approaching the boundary. The results presented in this paper are very useful in view of possible extensions to general Riemann surfaces. Moreover, they could also shed some light in the quantization of field theories on manifolds with constant curvature scalars in higher dimensions.

February 1995

## 1. INTRODUCTION

This paper treats the quantization of field theories on a two dimensional manifold with constant curvature scalar  $R$ . Positive values of  $R$  correspond to the topology of a sphere, whereas negative values correspond to hyperbolic geometries, like the Poincaré upper half plane or the Poincaré disk. Particular attention is devoted to the case of negative curvatures, which is relevant in many different contexts [1]–[6].

In the first part of the paper we deal with massive scalar fields coupled to  $R$  through a coupling constant  $\lambda$ . This model is interesting in itself. After a field redefinition, for example, the free equations of motion on the Poincaré upper half plane are equivalent to the Euler-Poisson-Darboux equations describing the propagation of waves in a polytropic gas. Moreover, since the manifolds with constant curvature are harmonic, it is always possible to reduce the equations of motion of the fields (also in the presence of self-interactions) to ordinary differential equations of the second order [7]. This is a valid alternative to the heat kernel techniques in computing the correlation functions. The free propagator is for instance solution of an equation of the hypergeometric type <sup>1</sup>, independently of the fact that  $R$  is positive or negative. In this sense it is possible to unify within the approach presented here the treatment of scalar fields on different topologies like the Poincaré disk or the sphere. The analogies between the two cases are however fortuitous and exist only at a formal level. As a matter of fact, the sphere is a manifold without boundary, whereas on the Poincaré disk or upper half plane proper boundary conditions should be imposed<sup>2</sup>. The latter are however determined by the geometry. For instance we show that, once we require that the propagator has only the physical singularity in the origin, automatically Dirichlet boundary conditions are chosen, in agreement with ref. [1].

In the second part of the paper we deal with the Schwinger model [8] on a Poincaré disk of radius  $\rho$ . Due to its physical relevance [9]–[12], the Schwinger model and its generalizations have been quantized on many different topologies [13]–[20] but not, until now, on an hyperbolic two dimensional manifold.

---

<sup>1</sup> We notice that similar equations have been found in ref. [4] for the massive fermionic fields.

<sup>2</sup> Let us remember that the in both cases of the disk and of the upper half plane the boundary does not belong to the manifold. Nevertheless, it is necessary to specify the limiting conditions with which the fields approach the boundary.

Conceptually, the computation of the anomaly in the presence of negative curvature presents many difficulties. The reason is that the Poincaré disk, like the upper half plane, is limited by a boundary. The latter does not belong to the manifold, however the behavior of the fields as they approach the boundary must be given. Moreover, the metric tensor becomes singular exactly on the boundary. Most of the mathematical and physical literature discussing the calculation of chiral determinants on curved space-times [21]–[23] does not treat this particular situation explicitly. An exception is however provided by ref. [24]. Fortunately, in order to derive the form of the anomaly in the present case one can also exploit a perturbative approach, avoiding mathematical complications [11]. This strategy will be adopted here, showing that there are no terms related to the boundary in the chiral determinant. Finally, in order to study the behavior of the correlation functions of the Schwinger model at short and long distances, we will use the propagator derived for the massive scalar fields in the first part of the paper. This task is made relatively easy by the fact that, as already remarked, this propagator is expressed in terms of hypergeometric functions. The 2-point  $\bar{\psi}\psi$  correlator is thoroughly studied in this way.

The presentation of the above discussed results is organized as follows. In Section 2 the theory of massive scalar fields coupled with the constant curvature scalar is quantized on the Poincaré upper half plane. The propagator is derived solving an hypergeometric equation. The behavior of the propagator and the choice of the boundary conditions are discussed in details. In Sections 3 and 4 are treated the cases of the sphere and of the Poincaré disk respectively. The Poincaré disk is equivalent to the upper half plane up to a conformal transformation. Finally, the Schwinger model on a Poincaré disk of radius  $\rho$  is investigated in Section 5. The fermionic propagator and the chiral determinants are computed using a perturbative approach. After bosonization, the effective theory becomes as in the flat case a free field theory of massive mesons and of free massless fermions. At the end of the Section the behavior of the correlation functions and in particular of the  $\bar{\psi}\psi$  2-point function are studied in details. All the formulas concerning hypergeometric functions used in this paper have been listed in Appendix A. In Appendix A we also show that the propagators of the scalar fields with the proper boundary conditions have only the physical singularity at the origin.

## 2. MASSIVE SCALAR FIELDS ON $\mathbf{H}^2$

Let us consider the functional:

$$S = \int_{\mathbf{H}^2} d^2x \sqrt{g} \left( \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{\mu^2}{2} \varphi^2 + (\lambda + 1) R \varphi \right) \quad (2.1)$$

This action describes a massive scalar field theory on the upper half plane  $\mathbf{H}^2$  parametrized by the coordinates  $x, y$ . The fields are coupled with the curvature scalar  $R$  through the real parameter  $\lambda$ . The metric on  $\mathbf{H}^2$  is Euclidean:

$$g_{\mu\nu} = \text{diag} \left( \frac{1}{y^2}, \frac{1}{y^2} \right) \quad (2.2)$$

Starting from this metric it is easy to see that the curvature scalar  $R$  amounts to a negative constant, which can be renormalized in such a way that  $R = -1$ .

At this point, it is convenient to introduce on  $\mathbf{H}^2$  the complex coordinates  $z = x + iy$  and  $\bar{z} = x - iy$ . Then the equation of motion  $\frac{\delta S}{\delta \varphi} = 0$  for the propagator  $G_{\mathbf{H}^2}(z, w) \equiv \langle \tilde{\varphi}(z, \bar{z}) \tilde{\varphi}(w, \bar{w}) \rangle$  reads as follows:

$$[(z - \bar{z})^2 \partial_z \partial_{\bar{z}} + \mu^2] G_{\mathbf{H}^2}(z, w) = \delta^{(2)}(z, w) - (\lambda + 1) R \quad (2.3)$$

Here we have exploited the identity  $\partial_z \partial_{\bar{z}} = \frac{1}{4}(\partial_1^2 + \partial_2^2)$  and the fact that, in complex coordinates, the components of the metric on  $\mathbf{H}^2$  are  $g^{z\bar{z}} = g^{\bar{z}z} = (\text{Im}[z])^2$ , where  $(\text{Im}[z])^2 = -\frac{1}{4}(z - \bar{z})^2$ . If  $\lambda = -1$ , eq. (2.3) becomes the usual equation

$$[-\Delta + \mu^2] G_{\mathbf{H}^2}(z, w) = \delta^{(2)}(z, w)$$

It is interesting to notice that if  $\lambda = 1$  eq. (2.3) is equivalent to the Euler-Poisson-Darboux equation describing the propagation of waves in a polytropic gas. After the change of variables:

$$\begin{cases} u = z & \bar{u} = -\bar{z} \\ v = w & \bar{v} = -\bar{w} \end{cases} \quad G_{\mathbf{H}^2}(u, v) = \left( \frac{u + \bar{u}}{v + \bar{v}} \right)^\beta f(u, v) \quad (2.4)$$

eq. (2.3) becomes in fact the Euler-Poisson-Darboux equation:

$$\partial_u \partial_{\bar{u}} f + \frac{\beta}{u + \bar{u}} (\partial_u f + \partial_{\bar{u}} f) = 0 \quad (2.5)$$

where  $\beta$  is defined by the relation:  $\beta(\beta - 1) = \mu^2$ .

To solve (2.3) we use the fact that a space with constant curvature is harmonic. Denoting with  $\Gamma$  the square of the geodetic distance between two points on  $\mathbf{H}^2$ , this means that  $\Delta\Gamma$  is a function of  $\Gamma$ . As a consequence, since the propagator must be a function of  $\Gamma$ , the equation of motion (2.3) becomes an ordinary differential equation of the second order in  $\Gamma$  [7].

In our case, it will be more convenient to choose instead of  $\Gamma$  the anharmonic ratio

$$X \equiv -\frac{(z-w)(\bar{z}-\bar{w})}{(z-\bar{z})(w-\bar{w})} \quad (2.6)$$

so that  $G_{\mathbf{H}^2}(z, w) \equiv G_{\mathbf{H}^2}(X)$ , with  $X \geq 0$  by construction. This is possible because the geodetic length  $\Gamma$  is a function of  $X$ . Considering now a generic function  $F(X)$  of  $X$  as an implicit function of  $z$  and  $\bar{z}$ , it is straightforward to prove the following relation:

$$(z - \bar{z})^2 \partial_z \partial_{\bar{z}} F(X) = -X(1 + X)F''(X) - (1 + 2X)F'(X) - XF'(X)\delta^{(2)}(z, w) \quad (2.7)$$

where the prime denotes the derivative in  $X$  and the Dirac delta function is defined as usual by:

$$\delta^{(2)}(z, w) = -\frac{1}{4\pi} g^{z\bar{z}} \partial_z \partial_{\bar{z}} \log(X) + \frac{R}{4\pi}$$

It is clear from the last term in eq. (2.7) that, in order to generate a Green function with the correct singularity in  $z = w$ , the behavior of  $F(X)$  should be logarithmic when  $X \sim 0$ , so that  $XF'(X) = 1 + O(X)$ . Substituting eq. (2.7) in eq. (2.3), we get a second order differential equation in  $G_{\mathbf{H}^2}(X)$  of the kind:

$$-X(X + 1)G_{\mathbf{H}^2}''(X) - (1 + 2X)G_{\mathbf{H}^2}'(X) + \mu^2 G_{\mathbf{H}^2}(X) = -(\lambda + 1)R \quad (2.8)$$

This is the desired final expression of the equations of motion.

At this point, we notice that eq. (2.8) is hypergeometric<sup>3</sup>. As a matter of fact, substituting  $y = -X$  and  $\eta(y) = G_{\mathbf{H}^2}(X)$ , eq. (2.8) becomes:

$$y(1 - y)\eta'' + (1 - 2y)\eta' + \mu^2\eta = -(\lambda + 1)R \quad (2.9)$$

---

<sup>3</sup> The properties of the hypergeometric functions that will be used here can be found for example in [25]–[26].

Now it is easy to construct the solutions of eq. (2.9). The only problem is to choose the physical boundary conditions when  $z$  and  $w$  approach the boundary of  $\mathbf{H}^2$  on the real line. Setting  $\beta = \frac{1}{2} - \sqrt{\frac{1}{4} + \mu^2}$ , we have that

$$\eta_1(y) = F(1 - \beta, \beta; 1|y) \quad (2.10)$$

is one of the two independent solutions of eq. (2.9) in the homogeneous case, i.e. when  $\lambda = 1$ . Here  $F(a, b; c|z)$  denotes the hypergeometric function

$$F(a, b; c|z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \quad (2.11)$$

with  $(a)_k = a(a+1)(a+2)\dots(a+k-1)$ . Clearly,  $\eta_1(y)$  is not singular in  $y = 0$  but has a logarithmic divergence in  $y = 1$ . The other independent solution of eq. (2.9) when  $\lambda = -1$  is given by:

$$\eta_2(y) = F(1 - \beta, \beta; 1|y) \int_{C_1}^y \frac{[t(1-t)]^{-1}}{[F(1 - \beta, \beta; 1|t)]^2} \quad (2.12)$$

$\eta_2(y)$  has the desired logarithmic singularity in  $y = 0$  with the correct sign, but diverges also in  $y = 1$  due to the presence of the factor  $F(1 - \beta, \beta; 1|y)$ . To avoid this problem, it is sufficient to take  $C_1 = 1$  in eq. (2.12). This choice can still be potentially dangerous, because apparently the integrand has a simple pole at  $t = 1$ . However, this singularity is integrable because of the denominator, which has the behavior  $[F(1 - \beta, \beta; 1|t)]^2 \sim \log^2(t)$  in a neighborhood of the point  $t = 1$ . Having the two independent solutions of the homogeneous problem, the final ingredient in order to solve eq. (2.9) completely is a particular solution. This is given by:

$$\bar{\eta}(y) = -R(\lambda + 1)F(1 - \beta, \beta; 1|y) \int_{C_1}^y dt \frac{[t(1-t)]^{-1}}{[F(1 - \beta, \beta; 1|t)]^2} \int_{C_2}^t ds F(1 - \beta, \beta; 1|s) \quad (2.13)$$

Again, taking  $C_1 = 1$  in the above equation, all the logarithmic singularities in the point  $X = 1$  disappear. In the same way, spurious singularities in  $X = 0$  can be eliminated choosing  $C_2 = 0$ .

At this point we are ready to construct the Green function of the massive scalar fields on  $\mathbf{H}^2$ . We choose that Green function in such a way that there is only a logarithmic singularity in  $X = 0$  on the half-line  $X \geq 0$ . This requirement completely removes the

arbitrariness in solving (2.8). Remembering that  $X = -y$ , the Green function satisfying eq. (2.8) with the desired pole structure is given by:

$$G_{\mathbf{H}^2}(X) = -\frac{1}{4\pi}F(1-\beta, \beta; 1| -X) \int_1^{-X} dt \frac{[t(1-t)]^{-1}}{[F(1-\beta, \beta; 1|t)]^2} - (1+\lambda)RF(1-\beta, \beta; 1| -X) \int_1^{-X} dt \frac{[t(1-t)]^{-1}]^{-1}}{[F(1-\beta, \beta; 1|t)]^2} \int_0^t ds F(1-\beta, \beta; 1|s) \quad (2.14)$$

From the above discussion and from the properties of the hypergeometric functions, it is clear that  $G_{\mathbf{H}^2}(X)$  is a well defined Green function, in particular when the integrand approaches the point  $t = 1$ . Moreover,  $G_{\mathbf{H}^2}(X)$  is regular on the half-line  $X > 0$ , but has a singularity of the kind  $G_{\mathbf{H}^2}(X) \sim -\frac{1}{4\pi}\log(-X)$  near  $X = 0^4$ . For the sake of completeness, however, a rigorous proof will be given in the appendix. Here we only notice that, using the series representation of the hypergeometric function (2.11), it is possible to show that

$$\int_0^t ds F(1-\beta, \beta; 1|s) = tF(1-\beta, \beta; 2|t) \quad (2.15)$$

Exploiting the above equation in (2.14), one can further simplify the expression of  $G_{\mathbf{H}^2}(X)$  eliminating the double integral in  $s$  and  $t$ . We remark also that in the limit  $\mu^2 = 0$  we have  $F(1, 0; 1|X) = 1$ . Therefore, when  $\lambda = -1$ , it is easy to see that  $G_{\mathbf{H}^2}(X)$  becomes, apart from an infinite constant, the usual scalar Green function of the massless case:

$$\lim_{\substack{\mu^2, \lambda+1 \rightarrow 0 \\ n=2}} G_{\mathbf{H}^2}(X) = -\frac{1}{4\pi}\log\left(-\frac{X}{1+X}\right) \quad (2.16)$$

Let us notice that the infinite constant is unavoidable in two dimensions when taking the small mass limit of the massive propagator. This is due to the unregularized infrared divergencies. Apart from that, one can easily show that eq. (2.16) is in agreement with ref. [1], where the massless scalar Green function has been computed on an hyperbolic disk. The latter is related to  $\mathbf{H}^2$  only by a conformal transformation (see below). Moreover, we can also see from eq. (2.16) and from the definition (2.6) of  $X$  that the function  $\log\left(-\frac{X}{1+X}\right)$  has not only a logarithmic singularity at  $z = w$ , but also a logarithmic singularity of the opposite sign at the *image* point  $z = \bar{w}$ , as pointed out in ref. [1]. This singularity, lying beyond the border  $y = 0$  of  $\mathbf{H}^2$ , is harmless and therefore we have:

$$-\frac{1}{4\pi}\Delta\log\left(-\frac{X}{1+X}\right) = \delta^{(2)}(z, w)$$

---

<sup>4</sup> Remember that the equation of motion (2.3) contains an overall negative sign.

In the limit of large  $X$ , instead, the propagator  $G_{\mathbf{H}^2}(X)$  has the following asymptotic behavior, obtained using the analytic continuation of the hypergeometric function at infinity:

$$\lim_{X \rightarrow \infty} G_{\mathbf{H}^2}(X) \sim c_1 X^{-\frac{1}{2} - \sqrt{\frac{1}{4} + \mu^2}} + c_2 R(\lambda + 1)$$

In the above equation  $c_1$  and  $c_2$  are constants depending on  $m^2$  and  $k^2$  which can be easily determined from the formulas given in the appendix. As it is possible to see, in absence of the inhomogeneous term  $(\lambda + 1)R$  the fall off of the propagator at infinity increases exponentially with the mass. We remember that, from our settings, the point  $X = \infty$  corresponds to the boundary  $\text{Im}z = 0$ . Therefore, the vanishing of  $G_{\mathbf{H}^2}(X)$  in  $X = \infty$  for  $\lambda = -1$  implies the choice of Dirichlet boundary conditions  $\varphi|_{\partial M} = 0$  on  $\mathbf{H}^2$ . The same boundary conditions are also satisfied by the right hand side of eq. (2.16). The fact that the Dirichlet boundary conditions are privileged on spaces with negative curvature has been pointed out also in ref. [1].

### 3. THE CASE OF THE SPHERE

Let us now consider the massive scalar fields on the sphere  $\mathbf{S}^2$ . We choose the metric  $g_{z\bar{z}} = \frac{1}{(1+z\bar{z})^2}$ , so that the scalar curvature, defined as  $R = g^{z\bar{z}}\partial_z\partial_{\bar{z}}\log(g_{z\bar{z}})$ , is strictly positive. The equation of motion analogous to (2.3) becomes in this case:

$$[-g^{z\bar{z}}\partial_z\partial_{\bar{z}} + \mu^2] G_{\mathbf{S}^2}(z, w) = \delta^{(2)}(z, w) - (\lambda + 1)R \quad (3.1)$$

In order to explicitly construct the fundamental solution of (3.1), it is convenient to seek for a Green function of the kind  $G_{\mathbf{S}^2}(z, w) \equiv G_{\mathbf{S}^2}(X)$ , where  $X$  is given now by:

$$X = \frac{(z - w)(\bar{z} - \bar{w})}{(1 + z\bar{z})(1 + w\bar{w})} \quad (3.2)$$

Considering a generic function  $F(X)$  as an implicit function of  $z, \bar{z}$ , one obtains the following result:

$$g^{z\bar{z}}\partial_z\partial_{\bar{z}}F(X) = X(1 - X)F''(X) + (1 - 2X)F'(X) + XF'(X)\delta^{(2)}(z, w) \quad (3.3)$$



As a consequence, the equation of motion (3.1) takes the form:

$$-X(1-X)G_{\mathbf{S}^2}''(X) - (1-2X)G_{\mathbf{S}^2}'(X) + \mu^2 G_{\mathbf{S}^2}(X) = \delta^{(2)}(z, w) - (\lambda + 1)R \quad (3.4)$$

This is exactly an hypergeometric equation and therefore the Green function can be formally derived as in the previous section. A difference with respect to the case of negative curvature is however provided by the fact that the parameter  $\beta$ , depending on the mass  $\mu^2$ , is now of the form:

$$\beta = \frac{1}{2} - \sqrt{\frac{1}{4} - \mu^2}$$

Therefore  $\beta$  is allowed to be complex if  $\mu^2 > 1/4$ . This feature has important consequences in the behavior of the propagator at large values of  $X$  as we will see below. After this remark, we give the explicit form of the propagator on  $\mathbf{S}^2$ , which can be computed exploiting the same strategy of the previous case:

$$\begin{aligned} G_{\mathbf{S}^2}(X) = & -\frac{1}{4\pi} F(1-\beta, \beta; 1|X) \int_1^X dt \frac{[t(1-t)]^{-1}}{[F(1-\beta, \beta; 1|t)]^2} + \\ & (1+\lambda)R F(1-\beta, \beta; 1|X) \int_1^X dt \frac{[t(1-t)]^{-1}]^{-1}}{[F(1-\beta, \beta; 1|t)]^2} \int_0^t ds F(1-\beta, \beta; 1|s) \end{aligned} \quad (3.5)$$

Also when  $\beta$  becomes complex, the fact that  $1-\beta = \bar{\beta}$  assures that  $G_{\mathbf{S}^2}(X)$  remains real as it should be. It is also easy to convince oneself that the only singularity of  $G_{\mathbf{S}^2}(X)$  occurs near the point  $X = 0$ , where  $G_{\mathbf{S}^2}(X) \sim \log(X)$ . Moreover, when  $\mu^2 = 0$ ,  $(1+\lambda)R = \frac{1}{4\pi}$  and  $n = 2$ ,  $G_{\mathbf{S}^2}(X)$  reduces to the usual Green function of the massless scalar fields on the sphere, i.e.:

$$\lim_{\substack{m, \lambda \rightarrow 0 \\ n=2}} G_{\mathbf{S}^2}(X) = \log(X)$$

Indeed, the right hand side fulfills the well known equation of the massless Green function on  $\mathbf{S}^2$ :

$$\Delta \log(X) = \delta^{(2)}(z, w) - \frac{1}{4\pi}$$

In the limit of large  $X$  the behavior of  $G_{\mathbf{S}^2}(X)$  is very different from that of  $G_{\mathbf{H}^2}(X)$ . As a matter of fact, we have at the leading order:

$$G_{\mathbf{S}^2}(X) \sim c_1' X^{-\frac{1}{2}} \log(X) + c_2'(\lambda - 1)$$

The decreasing at infinity of  $G_{\mathbf{S}^2}(X)$  without the inhomogeneous term proportional to  $R(\lambda + 1)$  is independent of the mass term, which contributes only to a complex phase in the coefficients  $c_1'$  and  $c_2'$ .

#### 4. THE CASE OF THE HYPERBOLIC DISK

Finally, we investigate the case in which the topology is given by a two dimensional disk  $\mathbf{D}^2$  of radius  $\rho$ , equipped with a metric of the kind  $g_{z\bar{z}}dzd\bar{z} = \frac{\rho^4 dzd\bar{z}}{(\rho^2 - z\bar{z})^2}$ . The scalar curvature on  $\mathbf{D}^2 \otimes \mathbf{R}^{n-2}$  is negative:  $R = -\frac{1}{\rho^2}$ . As a matter of fact, the hyperbolic disk  $\mathbf{D}^2$  can be obtained from  $\mathbf{H}^2$  after performing the conformal transformation  $z = J(\zeta)$ , where  $z \in \mathbf{D}^2$ ,  $\zeta \in \mathbf{H}^2$  and  $J(\zeta) = \frac{i\zeta + \rho}{\zeta + i\rho}$ . In this way  $\mathbf{D}^2$  provides a good test in order to confirm the results obtained in sections 2 and 3. Now the equation of motion of the massive scalar fields takes the form:

$$\left[ -\left(1 - \frac{z\bar{z}}{\rho^2}\right)^2 \partial_z \partial_{\bar{z}} + \mu^2 \right] G_{\mathbf{D}^2}(z, w) = \delta^{(2)}(z, w) - (\lambda + 1)R \quad (4.1)$$

To solve eq. (4.1) it is convenient to consider the ansatz  $G_{\mathbf{D}^2}(z, w) \equiv G_{\mathbf{D}^2}(X)$ , where

$$X = \frac{\rho^2(z - w)(\bar{z} - \bar{w})}{(\rho^2 - z\bar{z})(\rho^2 - w\bar{w})} \quad (4.2)$$

Repeating the same procedure followed in the previous section, we arrive at the following equation, valid for a generic differentiable function  $F(X)$ :

$$\left(1 - \frac{z\bar{z}}{\rho^2}\right)^2 \partial_z \partial_{\bar{z}} F(X) = X(1 + X)F''(X) + (1 + 2X)F'(X) + XF'(X)\delta^{(2)}(z, w)$$

As a consequence, eq. (4.1) becomes:

$$-X(1 + X)G_{\mathbf{D}^2}''(X) - (1 + 2X)G_{\mathbf{D}^2}'(X) + \mu^2 G_{\mathbf{D}^2}(X) = \delta^{(2)}(z, w) - (\lambda + 1)R \quad (4.3)$$

Performing the substitution  $y = -X$  as we did for  $\mathbf{H}^2$ , one gets the same hypergeometric equation (2.9). The desired propagator is therefore given by:

$$G_{\mathbf{D}^2}(X) = -\frac{1}{4\pi} F(1 - \beta, \beta; 1 | -X) \int_1^{-X} dt \frac{[t(1 - t)]^{-1}}{[F(1 - \beta, \beta; 1 | t)]^2} - \\ (1 + \lambda)RF(1 - \beta, \beta; 1 | -X) \int_1^{-X} dt \frac{[t(1 - t)]^{-1}}{[F(1 - \beta, \beta; 1 | t)]^2} \int_0^t ds F(1 - \beta, \beta; 1 | s) \quad (4.4)$$

As expected, this Green function coincides with  $G_{\mathbf{H}^2}(X)$  after performing the conformal transformation  $J : \mathbf{D}^2 \rightarrow \mathbf{H}^2$  introduced above. As a matter of fact, applying  $J$  to  $X$

as function of the variables  $z$  and  $w$  given by (4.2),  $G_{\mathbf{D}^2}(X)$  becomes exactly equal to  $G_{\mathbf{H}^2}(X')$ , where  $X'$  is the anharmonic ratio (2.6) in the variables  $\zeta = J(z)$  and  $\omega = J(w)$ .

## 5. THE SCHWINGER MODEL ON 2-D HYPERBOLIC GEOMETRIES

In this section we consider the massless Schwinger model (or two dimensional quantum electrodynamics QED<sub>2</sub>) on an hyperbolic disk  $\mathbf{D}^2$  of radius  $\rho$ . The extension to the Poincaré upper half plane  $\mathbf{H}^2$  can be achieved performing a conformal transformation and is straightforward. The action of the model is given by:

$$S_{\text{QED}_2} = \int_{\mathbf{D}^2} \frac{\rho^4 dx dy}{\rho^2 - |\vec{\xi}|^2} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi} e_\alpha^\mu \gamma^\alpha (\nabla_\mu + ieA_\mu) \psi \right] \quad (5.1)$$

In the above equation we have exploited the following notations. The metric is

$$g_{\mu\nu}(\vec{\xi}) = f(\vec{\xi}) \delta_{\mu\nu} \quad \vec{\xi} = (x, y) \quad (5.2)$$

with

$$f(\vec{\xi}) = \frac{\rho^4}{(\rho^2 - |\vec{\xi}|^2)^2} = \frac{\rho^4}{(\rho^2 - z\bar{z})^2}$$

The  $e_\alpha^\mu(\vec{\xi})$ , where  $\alpha, \mu = 0, 1$ , represent the vierbeins and  $\nabla_\mu$  denotes the covariant derivative acting on the fermions. Finally, the  $\gamma^\alpha$  are the usual two dimensional  $\gamma$ -matrices valid in the flat Euclidean space:

$$\begin{aligned} \gamma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \gamma^5 \equiv \gamma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

In local complex coordinates  $z$  and  $\bar{z}$ , the Dirac operator  $D_\alpha = e_\alpha^\mu \gamma^\alpha (\nabla_\mu + ieA_\mu)$  has the following components:

$$D_z = 2 \left[ f^{-\frac{1}{2}} (\partial_z - ieA_z) + \frac{\bar{z}}{2\rho^2} \right] \quad (5.3)$$

$$D_{\bar{z}} = -2 \left[ f^{-\frac{1}{2}} (\partial_{\bar{z}} - ieA_{\bar{z}}) + \frac{z}{2\rho^2} \right] \quad (5.4)$$

To simplify the action (5.1) it is convenient to decompose the gauge fields using the Hodge decomposition:

$$A_\mu = i\epsilon_{\mu\nu}\partial^\nu\varphi + \partial_\mu\chi \quad (5.5)$$

where  $\varphi$  and  $\chi$  are real scalar fields obeying the auxiliary conditions

$$\int_{\mathbf{D}^2} d^2\xi \sqrt{g}\varphi(\vec{\xi}) \neq 0 \quad \int_{\mathbf{D}^2} d^2\xi \sqrt{g}\chi(\vec{\xi}) \neq 0$$

Now we perform the following transformation on the fermionic fields:

$$\psi(\vec{\xi}) = e^{\imath e(\gamma_5\varphi(\vec{\xi})+\chi(\vec{\xi}))}\psi'(\vec{\xi}) \quad (5.6)$$

$$\bar{\psi}(\vec{\xi}) = \bar{\psi}'(\vec{\xi})e^{\imath e(\gamma_5\varphi(\vec{\xi})-\chi(\vec{\xi}))} \quad (5.7)$$

In this way, the massless Schwinger model (5.1) becomes a free field theory with an effective action containing an anomalous term. The latter is a pure quantum effect and can be explained in the path integral formalism by the noninvariance of the fermionic functional measure under the chiral transformation (5.6)-(5.7) [27]–[28]. In order to obtain the explicit expression of the anomalous term, we compute the determinant of the chiral operator  $\mathcal{D} = e_\alpha^\mu\gamma^\alpha(\nabla_\mu + ieA_\mu)$ . This calculation can be performed by means of heat kernel techniques. The case of the Poincarè disk is however exceptional, because the metric  $g_{\mu\nu}$  blows up exactly at the boundary  $\text{Re}(z) = 0$ . Most of the scientific literature on the subject assumes instead that the metric is finite [21]–[23]. To avoid this difficulty, we will compute the chiral determinant perturbatively. In **QED**<sub>2</sub>, in fact, the one loop radiative correction to the two point function of the gauge fields is sufficient in order to determine the exact result [11]. In other words, we have that:

$$\begin{aligned} & \text{Tr} \left\{ \ln \left[ \frac{\det(e_\alpha^\mu\gamma^\alpha(\nabla_\mu + ieA_\mu))}{\det(e_\alpha^\mu\gamma^\alpha\nabla_\mu)} \right] \right\} = \\ & \frac{e^2}{2} \int d^2\xi d^2\xi' \sqrt{g(\vec{\xi})}\sqrt{g(\vec{\xi}')} \langle \bar{\psi}(\vec{\xi})e_\alpha^\mu(\vec{\xi})\gamma^\alpha\psi(\vec{\xi})\bar{\psi}(\vec{\xi}')e_\beta^\nu(\vec{\xi}')\gamma^\beta\psi(\vec{\xi}') \rangle A_\mu(\vec{\xi})A_\nu(\vec{\xi}') \end{aligned} \quad (5.8)$$

The first ingredient needed in the calculation of the right hand side of eq. (5.8) is the free propagator of the fermionic fields. In complex notations, see e.g. [29], the two components of this propagator are given by:

$$\langle \bar{\psi}_\theta(z, \bar{z})\psi_\theta(w, \bar{w}) \rangle \equiv S_{\theta\theta}(z, w) \quad (5.9)$$

$$\langle \bar{\psi}_{\bar{\theta}}(z, \bar{z}) \psi_{\theta}(w, \bar{w}) \rangle = S_{\bar{\theta}\theta}(z, w) \quad (5.10)$$

and are characterized by the fact that, under a conformal transformation  $z \rightarrow z' = z'(z)$ , they transform in the following way:

$$\psi_{\theta}(z, \bar{z}) = \psi_{\theta'}(z', \bar{z}') \left( \frac{dz'}{dz} \right)^{\frac{1}{2}} \quad \bar{\psi}_{\theta}(z, \bar{z}) = \bar{\psi}_{\theta'}(z', \bar{z}') \left( \frac{dz'}{dz} \right)^{\frac{1}{2}}$$

and

$$\psi_{\bar{\theta}}(z, \bar{z}) = \psi_{\bar{\theta}'}(z', \bar{z}') \left( \frac{d\bar{z}'}{d\bar{z}} \right)^{\frac{1}{2}} \quad \bar{\psi}_{\bar{\theta}}(z, \bar{z}) = \bar{\psi}_{\bar{\theta}'}(z', \bar{z}') \left( \frac{d\bar{z}'}{d\bar{z}} \right)^{\frac{1}{2}}$$

The operators  $D_z$  and  $D_{\bar{z}}$  of eqs. (5.3) and (5.4) can be rewritten also in a simplified form, which will be convenient in future calculations:

$$D_z = 2f^{-\frac{3}{4}}(\partial_z - ieA_z)f^{\frac{1}{4}}$$

$$D_{\bar{z}} = 2f^{-\frac{3}{4}}(-\partial_{\bar{z}} + ieA_{\bar{z}})f^{\frac{1}{4}}$$

As a consequence we have that  $\nabla_z = 2f^{-\frac{3}{4}}(\partial_z)f^{\frac{1}{4}}$ ,  $\nabla_{\bar{z}} = 2f^{-\frac{3}{4}}(-\partial_{\bar{z}})f^{\frac{1}{4}}$  and the solutions of the free equations of motion

$$\nabla_z S_{\bar{\theta}\bar{\theta}}(z, w) = \delta_{\bar{z}z}(z, w) \quad \nabla_{\bar{z}} S_{\theta\theta}(z, w) = \delta_{\bar{z}z}(z, w)$$

are simply given by

$$S_{\theta\theta}(z, w) = \frac{1}{\pi} f^{-\frac{1}{4}}(z, \bar{z}) f^{-\frac{1}{4}}(w, \bar{w}) \frac{1}{z - w} \quad S_{\bar{\theta}\bar{\theta}}(z, w) = \frac{1}{\pi} f^{-\frac{1}{4}}(z, \bar{z}) f^{-\frac{1}{4}}(w, \bar{w}) \frac{1}{\bar{z} - \bar{w}} \quad (5.11)$$

This is the final expression of the propagator. Image charges are ruled out by the requirement of covariance under the  $\text{PSL}(2, \mathbf{R})$  group of transformations. For example, the propagator

$$S_{\theta\theta}(z, w) = \frac{1}{\pi} f^{-\frac{1}{4}}(z, \bar{z}) f^{-\frac{1}{4}}(w, \bar{w}) \left[ \frac{1}{z - w} - \frac{1}{\bar{z} - \bar{w}} \right]$$

satisfies the above equations of motion but does not transform according to the rule

$$S_{\theta\theta}(\gamma(z), \gamma(w)) = (cz + d)(cw + d)S_{\theta\theta}(z, w)$$

where

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is an element of  $\text{PSL}(2, \mathbf{R})$ . In real coordinates,  $\vec{\xi}$  and  $\vec{\xi}'$  we get:

$$S(\vec{\xi}, \vec{\xi}') = \frac{1}{2\pi} (f(\vec{\xi})f(\vec{\xi}'))^{-\frac{1}{4}} \gamma_\alpha \frac{(\xi^\alpha - \xi'^\alpha)}{|\vec{\xi} - \vec{\xi}'|^2} \quad (5.12)$$

Using the above propagator, we obtain the exact form of the chiral determinant:

$$\begin{aligned} \text{Tr} \left\{ \ln \left[ \frac{\det(e_\alpha^\mu \gamma^\alpha (\nabla_\mu + ieA_\mu))}{\det(e_\alpha^\mu \gamma^\alpha \nabla_\mu)} \right] \right\} = \\ -\frac{e^2}{2\pi} \int d^2\xi d^2\xi' \epsilon^{\mu\nu} \partial_{\nu(\xi)} A_\mu(\vec{\xi}) \log|\vec{\xi} - \vec{\xi}'|^2 \epsilon^{\rho\sigma} \partial_{\sigma(\xi')} A_\sigma(\vec{\xi}') \end{aligned} \quad (5.13)$$

Due to the presence of the partial derivatives in  $\vec{\xi}$  and  $\vec{\xi}'$ , the Green function  $\log|\vec{\xi} - \vec{\xi}'|^2$  in the right hand side of eq. (5.13) can be replaced by:

$$\frac{1}{\Delta} \equiv -\frac{1}{4\pi} \log \left[ \frac{|\vec{\xi} - \vec{\xi}'|^2}{\left(1 - \frac{\xi^2}{\rho^2}\right) \left(1 - \frac{\xi'^2}{\rho^2}\right)} \right] \quad (5.14)$$

where  $\Delta \equiv \sqrt{g} \partial_\mu \partial^\mu$  denotes the Laplacian. Thus eq. (5.13) represents the anomaly of the chiral Schwinger model in its usual form in curved space-times [29]. Decomposing the gauge fields by means of (5.5), we obtain the effective action of the Schwinger model in its final form:

$$S_{\text{QED}_2} = \int_{\mathbf{D}^2} d^2\xi \frac{\sqrt{g}}{2} \left[ \partial_\mu \varphi \left( \Delta - \frac{e^2}{2\pi} \right) \partial_\mu \varphi \right] - \int_{\mathbf{D}^2} d^2\xi \sqrt{g} [\bar{\psi}' \epsilon_\alpha^\mu \gamma^\alpha \nabla_\mu \psi'] \quad (5.15)$$

As we see, the  $\chi$  fields are completely decoupled and do not contribute as it happens in the flat case. The Green function of the scalar fields satisfies the equation:

$$\Delta \left( \Delta - \frac{e^2}{2\pi} \right) G(\vec{\xi}, \vec{\xi}') = \delta^{(2)}(\vec{\xi}, \vec{\xi}')$$

The solution can be easily obtained from eq. (4.4). After setting  $\lambda = -1$ ,  $X = \frac{\rho^2 |\vec{\xi} - \vec{\xi}'|^2}{(1 - \frac{\xi^2}{\rho^2})(1 - \frac{\xi'^2}{\rho^2})}$  and  $\mu^2 = \frac{e^2}{2\pi}$ , the result is:

$$G(\vec{\xi}, \vec{\xi}') = \frac{1}{m^2} \left[ G_{\mathbf{D}^2}(X)_{\lambda=-1} - G_{\mathbf{D}^2}(X)_{\lambda=-1, m=0} \right] \quad (5.16)$$

where  $G_{\mathbf{D}^2}(X)_{\lambda=-1, m=0} = \log \left( -\frac{X}{1+X} \right)$  and

$$G_{\mathbf{D}^2}(X)_{\lambda=-1} = -F(1 - \beta, \beta; 1 | -X) \int_1^{-X} dt \frac{[t(1-t)]^{-1}}{[F(1 - \beta, \beta; 1 | t)]^2}$$

Finally, the Green function of the fermionic fields  $\bar{\psi}$  and  $\psi$  has been already computed and it is given by eq. (5.12).

Now we are ready to derive the correlation functions of the Schwinger model. The most interesting correlators are those involving the original fermionic fields  $\bar{\psi}$  and  $\psi$  before of the transformation (5.6)-(5.7). As an example, we consider here the  $\bar{\psi}\psi$  2-point function:

$$\langle \bar{\psi}(\vec{\xi})\psi(\vec{\xi})\bar{\psi}(\vec{\xi}')\psi(\vec{\xi}') \rangle = \langle \bar{\psi}'(\vec{\xi})e^{ie\gamma_5\varphi(\vec{\xi})}\psi'(\vec{\xi})\bar{\psi}'(\vec{\xi}')e^{ie\gamma_5\varphi(\vec{\xi}')} \psi'(\vec{\xi}') \rangle \quad (5.17)$$

The propagators of the scalar fields  $\varphi(\vec{\xi})$  and of the fermionic fields  $\bar{\psi}$ ,  $\psi$ , are given in eqs. (5.16) and (5.12) respectively. Using these propagators, we obtain from (5.17):

$$\langle \bar{\psi}(\vec{\xi})\psi(\vec{\xi})\bar{\psi}(\vec{\xi}')\psi(\vec{\xi}') \rangle = \frac{1}{2\pi} (f(\vec{\xi})f(\vec{\xi}'))^{-\frac{1}{4}} \frac{\gamma_\alpha(\xi^\alpha - \xi'^\alpha)}{|\vec{\xi} - \vec{\xi}'|^2} e^{[-4e^2(G(\vec{\xi},\vec{\xi})+G(\vec{\xi}',\vec{\xi}')-2G(\vec{\xi},\vec{\xi}'))]} \quad (5.18)$$

where

$$G(\vec{\xi}, \vec{\xi}') = G(\vec{\xi}', \vec{\xi}') = \lim_{X \rightarrow 0} \frac{1}{m^2} \left[ G_{\mathbf{D}^2}(X)_{\lambda=1} - G_{\mathbf{D}^2}(X)_{\substack{\lambda=1 \\ \mu^2=0}} \right]$$

It is easy to see that this limit exists and is finite. Thus, the right hand side of eq. (5.18) has the expected behavior at short and long distances. When  $\xi \sim \xi'$ , in fact, it turns out that

$$G(\xi, \xi) + G(\xi', \xi') - 2G(\xi, \xi') \sim 0$$

and only the free fermionic propagator contributes. At long distances  $\xi \rightarrow \infty$ , we have instead that the four point function (5.18) converges to a finite number. To show this, let us for instance fix the value of  $\xi'$  and study the limit  $\xi \rightarrow \infty$  of (5.18). Clearly

$$\lim_{\xi \rightarrow \infty} \frac{f(\xi)^{-\frac{1}{2}}}{|\xi - \xi'|^2} = \text{const.}$$

Moreover, it is possible to check from eq. (4.2) that in the limit of large  $\xi$  the variable  $X$  is a finite constant, depending only on  $\xi'$ . As a consequence, also the exponent appearing in eq. (5.18) remains finite at large distances completing our proof. Finally one has to check the behavior of  $\langle \bar{\psi}(\vec{\xi})\psi(\vec{\xi})\bar{\psi}(\vec{\xi}')\psi(\vec{\xi}') \rangle$  near the boundary  $|z| = \rho$ , where  $X$  approaches infinity. Also in this case a straightforward calculation shows that the 2-point  $\bar{\psi}\psi$  function converges to a finite result.

## 6. CONCLUSIONS

In this paper we have quantized the massive scalar fields and the Schwinger Model on some relevant examples of two dimensional manifolds with negative and positive scalar curvature. The related correlation functions have been explicitly derived in terms of hypergeometric functions. Despite of the fact that there are many formal analogies between the case of the complex sphere and the Poincaré disk, it turns out that the behavior of the propagators in the infrared regime is very different, in particular if  $\mu^2 \leq \frac{1}{4}$ . Moreover, field theories on hyperbolic manifolds are complicated by the presence of the boundary. For instance, the calculation of the anomalous chiral determinant required in order to solve the Schwinger model is mathematically nontrivial on a Poincaré disk. In fact, this is a noncompact manifold with a metric tensor which diverges when approaching the boundary. The exact calculation of the chiral determinant of the Schwinger model could be computed here exploiting perturbation theory. Our result can shed some light also in similar problem arising in higher dimensions when hyperbolic manifolds with constant curvature are considered [2]–[3], [30].

Finally, it is the first time that the Schwinger model has been considered also on surfaces with negative curvatures like the Poincaré disk. Since the latter is equivalent to the upper half plane  $\mathbf{H}^2$  up to a conformal transformation, our results are a first step toward a complete solution of the Schwinger model on a closed and orientable Riemann surface  $\Sigma$ . As a matter of fact, a Riemann surface may be viewed as the ratio  $\Sigma = \mathbf{H}^2/\Gamma$ , where  $\Gamma$  is a Fuchsian group (see e. g. [31]). Until now, only the partition function and the generating functional of the fermionic currents have been nonperturbatively computed on a Riemann surface [20]. In this sense, our calculations can be useful in at least two ways. First of all, exploiting the theory of holomorphic forms on  $\mathbf{H}^2/\Gamma$ , it is possible to provide a closed expression for the propagator of the effective mesonic theory (5.15). In [20] the analogous of this Green function on  $\Sigma$  was in fact given only in terms of an infinite series, supposing that the mass term  $\mu^2 = \frac{e^2}{2\pi}$  is small. Moreover, since the free propagator of the fermionic fields exists in terms of theta functions [32], the idea of computing the chiral determinant by means of perturbation theory should work also on Riemann surfaces.



## 7. ACKNOWLEDGMENTS

The author wishes to thank G. Cognola, L. Vanzo and S. Zerbini for many valuable discussions and fruitful suggestions.

## Appendix A.

In this appendix we investigate the behavior of the hypergeometric functions  $\eta_1, \eta_2$  and  $\bar{\eta}$  discussed in section 3. The hypergeometric formulas are taken from [26]. The behavior in  $y = 0$  of  $\eta_2(y)$  is provided by the following expansion:

$$\eta_2(y) = F(1 - \beta, \beta; 1|y)\log(y) + \sum_{k=1}^{\infty} z^k \frac{(1 - \beta)_k (\beta)_k}{(k!)^2} [h(k) - h(0)]$$

where  $h(k) = \psi(1 - \beta + k) + \psi(\beta + k) - 2\psi(k + 1)$  and  $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ . Clearly, in  $y = 0$  the leading order term is  $\eta_2(y) \sim \log(y)$ .

In order to study the hypergeometric functions at the point  $y = 1$  the following expansion turns out to be very useful:

$$\begin{aligned} F(a, b; a + b + l|z) &= \frac{\Gamma(l)\Gamma(a + b + l)}{\Gamma(a + l)\Gamma(b + l)} \sum_{n=0}^{l-1} \frac{(a)_n (b)_n}{(1 - l)_n n!} (1 - z)^n + \\ &(1 - z)^l (-1)^l \frac{\Gamma(a + b + l)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} k_{n=0}^{\infty} \frac{(a + l)_n (b + l)_n}{n!(n + l)!} (1 - z)^n \end{aligned} \quad (\text{A.1})$$

where

$$k_n = \psi(n + 1) + \psi(n + 1 + l) - \psi(a + n + l) - \psi(b + n + l)$$

Moreover  $l$  is a nonnegative integer and the first sum should be set to zero if  $l = 0$ . Exploiting eq. (A.1) in the case  $a = 1 - \beta$ ,  $b = \beta$  and  $l = 0$ , it turns out that the leading order behavior of  $\eta_1(y)$  at  $y = 1$  is given by:

$$F(1 - \beta, \beta; 1|y) \sim -\frac{\Gamma(1)}{\Gamma(1 - \beta)\Gamma(\beta)} \log(1 - y) + \text{finite}$$

Consequently, in a neighborhood of  $y = 1$ , setting  $y = 1 - \epsilon$  with  $\epsilon$  small, we have:

$$\eta_2(1 - \epsilon) \sim \log(\epsilon) \int_1^{1-\epsilon} \frac{dt}{1-t} [\log(1-t)]^{-2}$$

This shows that the limit in  $y = 1$  of  $\eta_2(y)$  is well defined and amounts to a constant. This situation is improved in the case of the function (2.13). As a matter of fact the integrand in  $t$  appearing in the definition of  $\bar{\eta}(y)$  is the same as that of  $\eta_2(y)$  apart from the multiplication by the function  $tF(1 - \beta, \beta; 2|t)$ . However, this function vanishes in the neighborhood of the point  $t = 1$ , cancelling the singularity given by the presence of the factor  $(1 - t)^{-1}$ . This can be easily shown using eq. (A.1) with  $l = 2$ , which yields in  $t \sim 1$ :

$$tF(1 - \beta, \beta; 2|t) \sim (1 - t)\log(1 - t)$$

Finally, it is possible to investigate the behavior of the propagators near  $X = \infty$  by means of the following formula:

$$F(1 - \beta, \beta; 1|X) = \frac{\Gamma(1)\Gamma(2\beta - 1)}{\Gamma^2(\beta)} (-1)^{-1-\beta} z^{\beta-1} F(1 - \beta, 1 - \beta; 2 - 2\beta | \frac{1}{z}) +$$

$$\frac{\Gamma(1)\Gamma(1 - 2\beta)}{\Gamma^2(1 - \beta)} (-1)^\beta z^{-\beta} F(\beta, \beta; 2\beta | \frac{1}{z})$$

## References

- [1] C. G. Callan and F. Wilczek, *Nucl. Phys.* **B340** (1990), 366.
- [2] G. Cognola, K. Kirsten, L. Vanzo and S. Zerbini, *Phys. Rev.* **D49** (1994), 5307; A. A. Bytsenko, S. D. Odintsov and S. Zerbini, *Phys. Lett.* **336B** (1994), 355; G. Cognola and L. Vanzo, *Phys. Rev.* **D47** (1993), 4575.
- [3] S. Zerbini, *Lett. Math. Phys.* **27** (1993), 19.
- [4] J. Palmer, M. Beatty and C. A. Tracy, *Comm. Math. Phys.* **165** (1994), 97; R. Narayanan and C. A. Tracy, *Nucl. Phys.* **B340** (1990), 568.
- [5] P. Kleban and I. Vassilieva, *Phys. Rev. Lett.* **72** (1994), 3929; Z. Haba, *Int. Jour. Mod. Phys.* **A4** (1989), 267; A. Comtet, *Ann. Phys.* **173** (1987), 173.
- [6] C. Grosche, *Fortsch. Phys.* **38** (1990), 531; J. Bolte and C. Grosche, *Comm. Math. Phys.* **163** (1994), 217.
- [7] F. G. Friedlander, *The wave Equation on a Curved Space-Time*, Cambridge University Press, Cambridge, London, New York, Melbourne 1975.
- [8] J. Schwinger, *Phys. Rev.* **128** (1962), 2425.
- [9] S. Coleman, *Phys. Rev.* **D11** (1975), 3026; S. Donaldson, *J. Diff. Geom.* **18** (1983), 269; R. Jackiw and R. Rajaraman, *Phys. Rev. Lett.* **54** (1985), 1219; L. Faddeev and S. Shatashvili, *Phys. Lett.* **183B** (1987), 311; T. P. Killingback, *Phys. Lett.* **223B** (1989) 357; A. I. Bocharek and M. E. Schaposhnik, *Mod. Phys. Lett.* **A2** (1987), 991; J. Kripfganz and A. Ringwald, *Mod. Phys. Lett.* **A5** (1990), 675.
- [10] J. H. Lowenstein, J. A. Swieca, *Ann. Phys.* **68** (1961), 172; A. Z. Capri and R. Ferrari, *Nuovo Cim.* **62A** (1981), 273; *Journ. Math. Phys.* **25** (1983), 141; G. Morchio, D. Pierotti and F. Strocchi, *Ann. Phys.* **188** (1988), 217; A. K. Raina and G. Wanders, *Ann. of Phys.* **132** (1981), 404.
- [11] E. Abdalla, M. C. B. Abdalla and K. Rothe, *Nonperturbative Methods in 2 Dimensional Quantum Field Theory*, World Scientific, Singapore, 1991.
- [12] N. Manton, *Ann. Phys.* **159** (1985), 220; J. E. Hetrick and Y. Hosotani, *Phys. Rev.* **38** (1988), 2621.
- [13] C. Jayewardena, *Helv. Phys. Acta* **61** (1988), 636.
- [14] A. Bassetto and L. Griguolo, *The Generalized Chiral Schwinger Model on the Two Sphere*, Preprint DFPD-94-TH-62, hep-th/9411229.
- [15] H. Joos and S. I. Azakov, *The Geometric Schwinger Model on the Torus. 2.*, Preprint DESY-94-142, H. Joos, *Nucl. phys.* **B17** (Proc. Suppl.) (1990), 704; *Helv. Phys. Acta* **63** (1990), 670.
- [16] S. G. Rajeev, *Phys. Lett.* **212B** (1988), 203, K. S. Gupta, R. J. Henderson, S. G. Rajeev and O. T. Turgut, *Jour. Math. Phys.* **35** (1994), 3845.
- [17] I. Sachs and A. Wipf, *Helv. Phys. Acta* **65** (1992), 653; *Phys. Lett.* **B326** (1994), 105.
- [18] J. Barcelos-Neto and A. Das, *Phys. Rev.* **D33** (1986), 2262.

- [19] F. Ferrari, *Class. Q. Grav.* **10** (1993), 1065; *Helv. Phys. Acta* **67** (1994), 702.
- [20] F. Ferrari, *On the Schwinger Model on Riemann Surfaces*, to appear in *Nucl. Phys. B*, hep-th/9310055.
- [21] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Math. Proc. Camb. Phil. Soc.* **77** (1975), 43; M. F. Atiyah, R. Bott and V. K. Patodi, *Inventiones Math.* **19** (1973), 279; R. T. Seeley, *Am. Math. Soc. Proc. Symp. Pure Math.* **10** (1967), 288; P. B. Gilkey, *Proc. Symp. Pure Math.* **27** (1973); *Jour. Diff. Geom.* **10** (1975), 601.
- [22] S. W. Hawking, *Comm. Math. Phys.* **55** (1977), 133; D. M. McAvity and H. Osborn, *Class. Q. Grav.* **8** (1991), 603; H. Leutwyler, *Phys. Lett.* **152B** (1985), 78; R. E. Gamboa Saravi, M. A. Muschietti and J. Solomin, *Comm. Math. Phys.* **89** (1983), 363; L. Alvarez-Gaumé and P. Ginsparg, *Nucl. Phys.* **B243** (1984), 449; O. Alvarez, I. M. Singer and B. Zumino, *Comm. Math. Phys.* **96** (1984), 409; L. Bonora, M. Bregola and P. Pasti, *Phys. Rev. D* **31** (1985), 2645; A. Andrianov and L. Bonora, *Nucl. Phys.* **B233** (1984), 232; *ibid.* 247; L. Bonora, P. Cotta-Ramusino and C. Reina, *Phys. Lett.* **126B** (1983), 305.
- [23] A. V. Mishchenko and Yu. A. Sitenko, *Ann. Phys.* **218** (1992), 199
- [24] A. J. Niemi and G. W. Semenoff, *Nucl. Phys.* **B269** (1986), 131.
- [25] E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen*, I (*Gewöhnliche Differentialgleichungen*) (7th ed.), Akademische Verlagsgesellschaft, Leipzig (1961).
- [26] A. Erdélyi and H. Bateman, *Higher transcendental functions*, Vol. I, Mc Graw-Hill Book Company Inc., New York 1953.
- [27] K. Fujikawa, *Phys. Rev. Lett.* **42** (1979), 1195; K. Fujikawa, *Phys. Rev. D* **21** (1980), 2848; K. Fujikawa, *Phys. Rev. D* **22** (1980), 1499(E).
- [28] J. Barcelos-Neto and Ashok Das, *Phys. Rev. D* **33** (1986), 2262.
- [29] L. Alvarez-Gaumé, G. Moore and C. Vafa, *Comm. Math. Phys.* **106** (1986), 1.
- [30] R. Camporesi, *Phys. Rep.* **196** (1990), 1.
- [31] E. D'Hoker and D. H. Phong, *Rev. Mod. Phys.* **60** (4) (1988), 917.
- [32] E. Verlinde and H. Verlinde, *Nucl. Phys.* **B288** (1987), 357; M. Bonini and R. Iengo, *Int. Jour. Mod. Phys. A* **3** (1988), 841.